1. Preliminaries

1.1) Dimensions and Units

Defn: Dimensions:
Every physical quantity has a <u>physical dimension</u>.
<u>Notation</u>: $[u]$ represents physical dimensions of u.

<u>Example</u>:

$$\left.\begin{array}{ll} [\text{Linear position}] = L & (\underline{\text{length}}) \\ [\text{time}] = T & (\underline{\text{time}}) \\ [\text{mass}] = M & (\underline{\text{mass}}) \end{array}\right\} \text{The 3 fundemental dimensions}$$

$[\text{velocity}] = L/T$ (length $\div$ time)

$[\text{area}] = L^2$ (length squared).

<u>Note</u>: Real numbers are dimensionless quantities. The dimension of a real number is 1

$$\forall x \in \mathbb{R}\ ;\ [x] = 1$$

Defn: __Units__

Any physical quantity that can be measured needs a unit of measurement.

__Example:__

Linear Position/distance/length can be measured by units such as metres, inches, miles etc.

__S.I Units:__

This module uses S.I Units for measurements:

length: meters (m)

time: seconds (s)

mass: Kilograms (kg)

Physical dimensions follow certain rules:
The following rules are axioms:

Axioms:

1) Real numbers are dimensionless, i.e. real numbers have a dimension of 1

2) For any two physical quantities A and B,

$$A = B \implies [A] = [B]$$

We cannot compare two quantities with different physical dimensions. (contrapositive)

$$[A] \neq [B] \implies A \neq B$$

3) For any 3 physical quantities A, B and C,

$$A \pm B = C \implies [A] = [B] = [C]$$

4) For any 2 physical quantities A and B,

$$[A.B] = [A].[B] \quad \text{and} \quad [A/B] = [A]/[B]$$

Note: Rules 1-4 imply:

$$\forall \lambda \in \mathbb{R}, \quad [\lambda A] = [\lambda].[A] = 1.[A] = [A]$$

$$\Rightarrow \quad [\lambda A] = [A]$$

Example problem 1:

Let $x(t)$ be the distance a car moved from a fixed point on the road.

$$x(t) = \alpha t^2 + \beta t + \gamma e^{-\lambda t}$$

for some constants $\alpha, \beta, \gamma, \lambda$.
what are the physical dimensions of these constants?

Solution:

From rule (3), it follows that

$$[\alpha t^2] = [\beta t] = [\gamma e^{-\lambda t}] = [x] = L$$

$$[\alpha t^2] = [\alpha t^2] = [\alpha][t]^2 = \alpha T^2 = L$$

$$\Rightarrow \boxed{\alpha = L/T^2}$$

$[\beta t] = L \Rightarrow [\beta].[t] = L \Rightarrow [\beta].T = L$

$\Rightarrow \boxed{[\beta] = L/T}$

$[\gamma e^{-\lambda t}] = [\gamma].[e^{-\lambda t}] = [\gamma].1 = L$

$\Rightarrow \boxed{[\gamma] = L}$

$[-\lambda t] = 1 \Rightarrow [\lambda].[t] = 1 \Rightarrow [\lambda].T = 1$

$\Rightarrow \boxed{[\lambda] = 1/T}$

Argument for exponential function must be dimensionless

$\forall x$, e^x is a real number $\Rightarrow [e^x] = 1$

1.2) Kinematics in 1D.

Co-ordinate system:

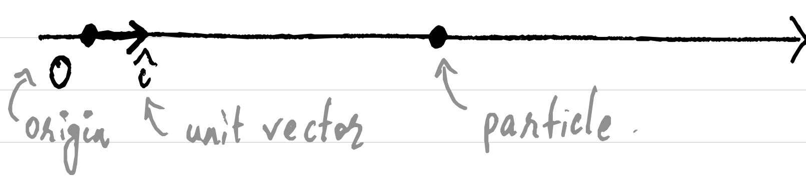
Consider a particle going on a straight line:

The following simplification is used:

Any object whose motion is studied will be represented as a point.

To describe motion mathematically, we need a co-ordinate system

1) First we need to chose an arbitrary point on the line and call it the origin.

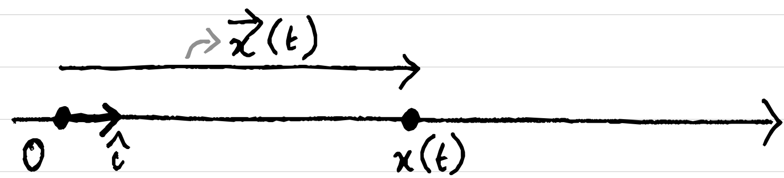


2) Second we choose the co-ordinate axis along the line of motion and chose its positive direction.

3) Finally we define a unit vector $\hat{\imath}$ whose direction is the same as the positive direction of the co-ordinate axis.

$\hat{\imath}$ is a unit vector $\Rightarrow |\hat{\imath}| = 1$.

Defn: Position vector:
Motion of a particle is described by a <u>position vector</u> $\vec{x}(t)$.



If we know the <u>position vector</u> of the particle, we know everything about the particle.

With the help of the unit vector $\hat{\imath}$, the position vector $\vec{x}(t)$ can be represented by the form

$$\vec{x}(t) = x(t)\,\hat{\imath}$$

where $x(t)$ is a <u>scalar function</u> called the <u>co-ordinate</u> or <u>position</u> or the <u>component</u> of the position vector of the particle.

Note: The co-ordinate $x(t)$ can be positive or negative or zero.

- If $x(t) > 0$ then direction of position vector is same as the positive direction of the co-ordinate axis.

- If $x(t) < 0$ then the direction of the position vector is opposite to the positive direction of the co-ordinate axis.

Remark:
It is important to distinguish vector quantities from scalar ones. The following notation will be used:

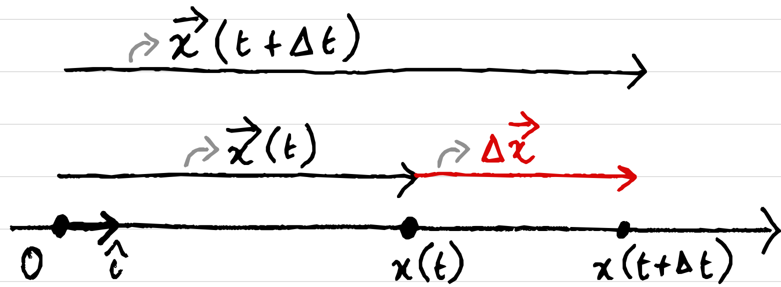
- $\vec{v}$ will represent a vector quantity
- v will represent a scalar quantity.

Defn: Displacement Vector

Consider now two consecutive moments of time, t and $t+\Delta t$.

We define <u>displacement vector</u> on the interval $[t, t+\Delta t]$ as

$$\Delta\vec{x} = \vec{x}(t+\Delta t) - \vec{x}(t)$$



Again using the unit vector $\hat{\imath}$, we can write the displacement vector in the form:

$$\Delta\vec{x} = \vec{x}(t+\Delta t) + \vec{x}(t)$$

$$= \big(x(t+\Delta t) - x(t)\big)\hat{\imath}$$

$$= \Delta x\,\hat{\imath}$$

$$\Rightarrow \quad \Delta\vec{x} = \Delta x\,\hat{\imath}$$

where $\Delta\vec{x}$ is the component of the displacement vector.
It is a scalar quantity and can be positive, negative or zero.

- If $\Delta x > 0$ then the particle has moved in the positive direction of the co-ordinate axis, over time interval $[t, t+\Delta t]$

- If $\Delta x < 0$ then the particle has moved in the negative direction of the co-ordinate axis, over time interval $[t, t+\Delta t]$

Defn: **Average Velocity:**
Average Velocity on the interval $[t, t+\Delta t]$ is defined as

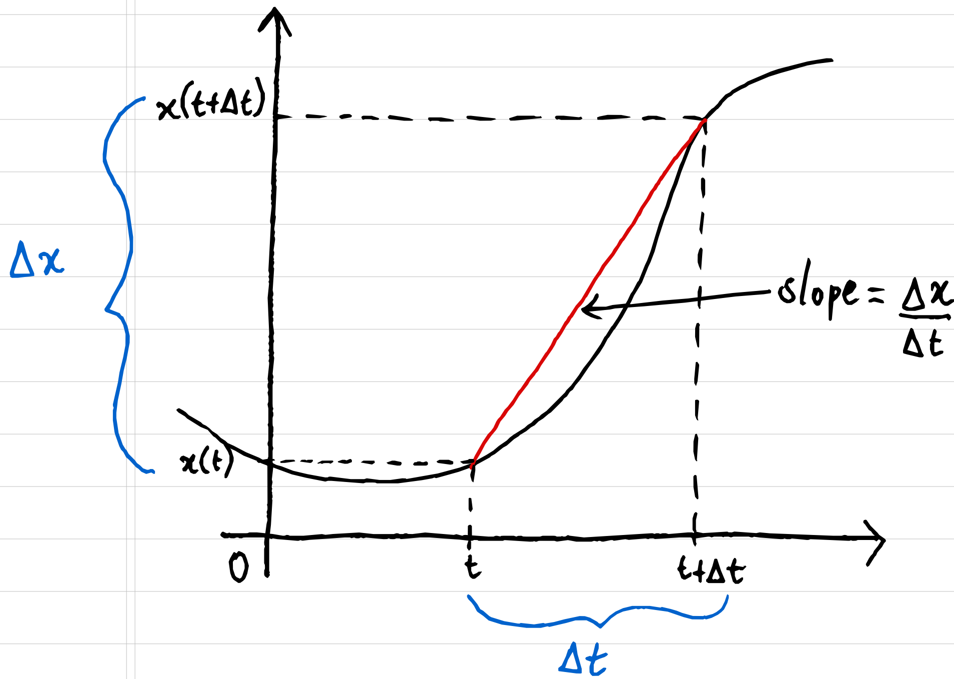
$$\vec{v}_A = \frac{\Delta\vec{x}}{\Delta t} = \frac{\Delta x}{\Delta t}\hat{\imath} = v_A(t)\hat{\imath}$$

where $v_A(t)$ is the component of the average velocity vector (or simply the average velocity)

Scalar $v_A(t)$ can be positive, negative or zero.

Geometric meaning: (average velocity)

The geometric meaning of the average velocity $v_A(t)$ is the slope of the line connecting two points of the graph $x(t)$ corresponding to t and $t+\Delta t$.



Defn: Instantaneous Velocity:

Instantaneous velocity at time t_1 is defined as

$$\vec{v}(t_1) = \lim_{\Delta t \to 0} \frac{\vec{x}(t_1+\Delta t) - \vec{x}(t)}{\Delta t}$$

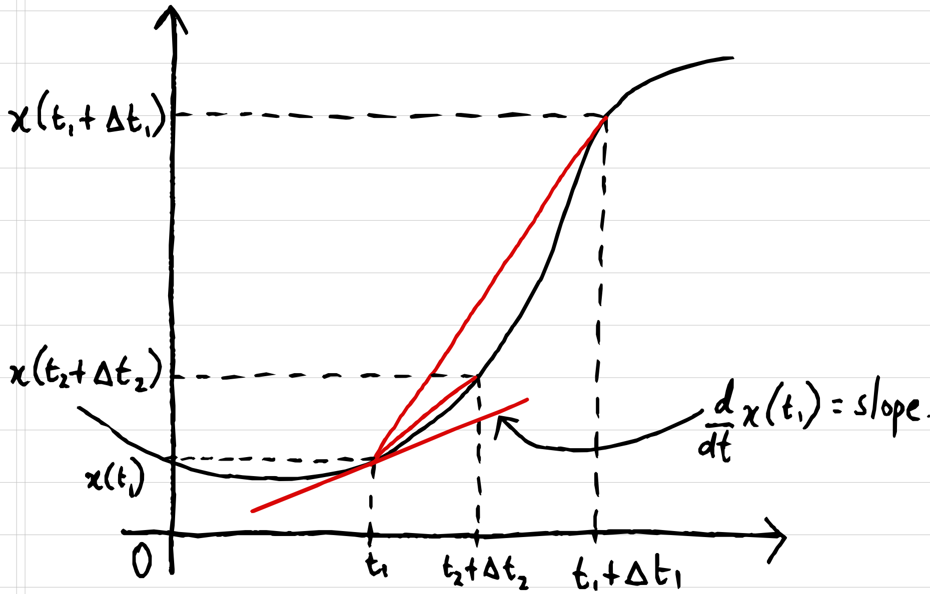
$$= \lim_{\Delta t \to 0} \frac{x(t_1+\Delta t) - x(t)}{\Delta t}\,\hat{i}$$

$$= \frac{dx(t_1)}{dt}\,\hat{i}$$

$$= v(t_1)\,\hat{i}$$

Geometric meaning: (instantaneous velocity)
The geometric meaning of the instantaneous velocity $v(t_1)$ is the slope of the tangent line to the graph of $x(t)$ at $t = t_1$.

Diagramatic representation is shown on next page.



We have defined velocity at time t_1. The same can be done for any moment in time.
So, the instantaneous velocity at any time t is

$$\vec{v}(t) = \frac{d\vec{x}}{dt} = \frac{d}{dt}x(t)\hat{\imath} = v(t)\hat{\imath}.$$

Physical meaning of velocity:
the velocity of the particle is the rate of change of position.

Defn: Instantaneous Acceleration:

The instantaneous acceleration $\vec{a}(t)$ can be defined in a similar manner as

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d}{dt} v(t)\hat{i} = a(t)\hat{i}$$

or equivalently,

$$\vec{a}(t) = \frac{d^2\vec{x}}{dt^2} = \frac{d^2}{dt^2} x(t)\hat{i} = a(t)\hat{i}$$

Also if $\vec{v} = v(x(t))\hat{i}$ then

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{dv}{dt}\hat{i}$$

$$= \frac{dv}{dx} \cdot \frac{dx}{dt}\hat{i} \qquad \text{[chain rule]}$$

$$= \frac{dv}{dx} \cdot v\hat{i}.$$

<u>Physical meaning of acceleration:</u>

The acceleration of the particle is rate of change of velocity

Example problem 2:

The position t of the particle moving on a straight line is given by

$$x(t) = \alpha t^2 + \beta t + \gamma e^{-\lambda t}$$

for some constants $\alpha, \beta, \gamma, \lambda$.
Find its velocity and acceleration check whether answer is dimensionally correct.

Solution:

The velocity is

$$v(t) = \frac{dx(t)}{dt} = \frac{d}{dt}\left(\alpha t^2 + \beta t + \gamma e^{-\lambda t}\right)$$

$$= 2\alpha t + \beta - \lambda \gamma e^{-\lambda t}$$

$$a(t) = \frac{dv(t)}{dt} = \frac{d}{dt}\left(2\alpha t + \beta - \lambda \gamma e^{-\lambda t}\right)$$

$$= 2\alpha + \lambda^2 \gamma e^{-\lambda t}$$

We shall only check that the answer for the acceleration is dimensionally correct. We need to show

$$[a(t)] = [2\alpha + \lambda^2 \gamma e^{-\lambda t}]$$

On the LHS we have

$$[a] = \left[\frac{dv}{dt}\right] = \frac{[v]}{[t]} = \frac{L/T}{T} = \frac{L}{T^2}$$

(using axioms 3 and 4)

On the RHS we have

$$[\alpha] = [\lambda^2][\gamma] = [a] = \frac{L}{T^2}$$

(using axioms 1, 3, 4)

From example 1,

$$[\alpha] = \frac{L}{T^2} \qquad [\lambda] = \frac{1}{T} \qquad [\gamma] = L$$

1.3) Motion on an Inclined plane

Physical laws are revealed as a result of observations and experiments

1.4) Ordinary Differential Equations

<u>Notation</u>: From now on, we shall use the following notation:

$$\boxed{\dot{x}(t) = \frac{dx(t)}{dt}}, \quad \boxed{\ddot{x} = \frac{d^2x(t)}{dt}},$$

$$\boxed{\vec{\dot{x}}(t) = \frac{d\vec{x}(t)}{dt} = \frac{dx(t)}{dt}\hat{\imath} = \dot{x}(t)\hat{\imath}},$$

$$\boxed{\vec{v}(t) = \vec{\dot{x}}(t) = \dot{x}(t)\hat{\imath}}$$

$$\boxed{\vec{\dot{v}}(t) = \frac{d\vec{v}(t)}{dt} = \frac{dv(t)}{dt}\hat{\imath} = \dot{v}(t)\hat{\imath}}$$

$$\boxed{\vec{\dot{v}}(t) = \frac{d\vec{v}(t)}{dt} = \frac{d^2\vec{x}(t)}{dt^2} = \vec{\ddot{x}}(t)}$$

$$\boxed{\vec{\dot{v}}(t) = \vec{a}(t) = \vec{\ddot{x}}(t)}$$

$$\boxed{a(t) = \ddot{x}(t) = \dot{v}(t)}$$

Defn: 1^{st} order ODE

Most first order ODE can be written as

$$F(x, \dot{x}, t) = 0$$

Here $x(t)$ is called the <u>dependant variable</u> and is an unknown of the <u>independant variable</u> t

<u>Example problem 3:</u>

The general solution of the ODE

$$\dot{x} + \lambda x = 0 \qquad (\lambda \in \mathbb{R} \text{ constant})$$

$$\Rightarrow \frac{dx}{dt} = -\lambda x$$

$$\Rightarrow \frac{1}{x} dx = -\lambda dt$$

$$\Rightarrow \int \frac{1}{x} dx = \int -\lambda dt$$

$$\Rightarrow \ln(x) = -\lambda t + c$$

$$\Rightarrow x = e^{-\lambda t + c}$$

$$\Rightarrow x(t) = e^{c} e^{-\lambda t}$$

$$\Rightarrow x(t) = Ce^{-\lambda t}$$

C is any arbitrary constant.

The general solution of any first order ODE contains one arbitrary constant.

In order to obtain a unique solution, we need to specify an initial condition, e.x,

$$x(0) = x_0 \qquad \text{(for some given constant } x_0\text{)}$$

Linear:

If the function $F(x, \dot{x}, t)$ is linear in the unknown function $x(t)$ and its derivative $\dot{x}$ then the ODE is said to be linear.

The most general first order linear ODE has form

$$\boxed{\dot{x} + A(t)x = f(t)} \qquad \text{eq } (*1)$$

where $A(t)$ and $f(t)$ are given functions

Homogeneous and inhomogeneous:

The above first order linear ODE (eq $(*1)$) is homogeneous if $f(t) = 0$ for all t

It is inhomogeneous if $f(t) \neq 0 \ \forall t$.

A very important and useful result:

The general solution of a linear inhomogeneous ODE is the sum of any particular solution of the inhomogeneous equation and the general solution of the homogeneous equation.

Defn: 2nd Order ODE

The most general second order ODE has form

$$F(x, \dot{x}, \ddot{x}, t) = 0$$

Example problem 4:

The general solution of the second order ODE

$$\ddot{x} - \lambda^2 x = 0$$

Ansatz: Assume solution of form

$$x = e^{at}$$

We get Auxillary equation

$$a^2 - \lambda^2 = 0 \Rightarrow (a-\lambda)(a+\lambda) =$$

$$\Rightarrow a = \pm\lambda$$

General solution is

$$x(t) = Ae^{\lambda t} + Be^{-\lambda t}$$

The general solution of any 2nd order ODE contains 2 arbitrary constants. To obtain a unique solution, we need 2 initial conditions eg

$$x(0) = x_0, \quad \dot{x}(0) = V_0$$

The most general linear 2nd order ODE looks like this

$$\boxed{\ddot{x} + A(t)\dot{x} + B(t)x = f(t)} \quad \text{eq}(*2)$$

where $A(t)$, $B(t)$, $f(t)$ are given functions.

Homogeneous and inhomogeneous

The above second order linear ODE is homogeneous if $f(t) = 0 \quad \forall t$

It is inhomogeneous if $f(t) \neq 0 \; \forall t$

A very important and useful result:

The general solution of a linear inhomogeneous ODE is the sum of any particular solution of the inhomogeneous equation and the general solution of the homogeneous equation.

Linear homogeneous 2nd order equation with constant coefficients

$$\ddot{x} + A\dot{x} + Bx = 0$$

can be solved by using an <u>Ansatz</u>: assuming that the solution has form

$$x(t) = e^{\lambda t}, \quad \lambda \in \mathbb{R}.$$

Substitution yields Auxillary equation

$$\lambda^2 + A\lambda + B = 0$$

<u>There are 3 possible cases:</u>

- If it has two <u>distinct real roots</u> λ_1 and λ_2 then the general solution of the ODE is

$$x(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$$

- If it has 2 complex conjugate roots, $\alpha + i\beta$ and $\alpha - i\beta$
  Then the general solution is given by

$$x(t) = c_1 e^{\alpha t}\sin(\beta t) + c_2 e^{\alpha t}\cos(\beta t)$$

- If it has one double root λ_0 then the general solution has the form

$$x(t) = Ae^{\lambda_0 t} + Bte^{\lambda_0 t}$$